JOURNAL OF APPROXIMATION THEORY 41, 279–290 (1984)

Optimal Interpolation with Incomplete Polynomials

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Communicated by R. Bojanic

Received July 15, 1983; revised October 24, 1983

INTRODUCTION

The "equioscillation" conditions of Bernstein [1] and Erdös [5] were originally conjectured to characterize optimal Lagrange interpolation. These "equioscillation" conjectures have been upheld as theorems in their original form and in other contexts as may be seen in [2, 3, 6, 7]. In all of these cases the proofs have used the same basic components, raising the possibility that these established cases are indeed particular manifestations of a general phenomenon. We therefore take the opportunity to formulate a conjecture about a general problem. The special case solved here will serve to illustrate some of the difficulties faced in an attempt to solve the more general problem.

Let Y be an $n+1$-dimensional subspace of $C[a, b]$, which is spanned by a Tchebycheff system. For given nodes $t_0, \ldots, t_n$ such that

$$a = t_0 < t_1 < \cdots < t_n = b,$$

let $\{y_0, \ldots, y_n\}$ be the basis for Y such that

$$y_i(t_j) = \delta_{ij} \qquad \text{(Kronecker delta)}.$$

Then an interpolation operator

$$L: C[a, b] \to Y$$

is defined by

$$Lf = \sum_{i=0}^{n} f(t_i)\, y_i,$$

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and we have

$$\|L\| = \left\| \sum_{i=0}^{n} |y_i| \right\|.$$

Noting that

$$\sum_{i=0}^{n} |y_i(t_j)| = 1 \qquad \text{for} \quad j \in \{0, ..., n\},$$

and setting

$$\lambda_j = \max_{t \in [t_{j-1}, t_j]} \sum_{i=0}^{n} |y_i(t)| \qquad \text{for} \quad j \in \{1, ..., n\},$$

we observe that

$$\|L\| = \max\{\lambda_1, ..., \lambda_n\},$$

which depends upon the choice of the nodes $t_1, ..., t_{n-1}$.

The Bernstein condition previously mentioned is that $\|L\|$ is minimal when $\lambda_1 = \cdots = \lambda_n$. The Erdös condition is that the norm of an L which is minimal always lies between $\min\{\lambda_1, ..., \lambda_n\}$ and $\max\{\lambda_1, ..., \lambda_n\}$.

It seems plausible, based upon several known cases, and upon [4], in which it is shown that the nodes $t_1, ..., t_{n-1}$ always exist such that $\lambda_1 = \cdots = \lambda_n$, to conjecture that optimal interpolation satisfies the Bernstein and Erdös conditions whenever the space Y contains the constant function and $n \geqslant 2$. Further evidence for this conjecture is that it is always true when $n = 2$.

All methods used so far in the proof of Bernstein and Erdös conjectures for special Tchebycheff systems $\{y_0, ..., y_n\}$ rely on the fact that the function

$$\sum_{i=0}^{n} |y_i(t)|$$

assumes maximum values $\lambda_1, ..., \lambda_n$ inside the corresponding intervals $[t_{j-1}, t_j]$, $j = 1, ..., n$.

If one assumes that Y is the span of an extended Tchebycheff system, or contains the constant function and if $n \geqslant 2$, then each λ_i must be achieved at a point in (t_{i-1}, t_i), and moreover $\lambda_i > 1$. In this context, the Bernstein and Erdös conditions, if demonstrated, become a non-trivial and meaningful characterization of optimality.

If, however, the space Y does not contain the constant functions, it is possible that an interpolation into Y may exist for which some or all of the maxima λ_i are achieved at nodes. For example, $1-t^2, t^3, t^4$ is a Tchebycheff system on the interval $[-1, 1]$, and on the nodes $t_0=-1$, $t_1=0$, $t_2=1$ we have

$$y_0(t)=\frac{t^3(t-1)}{2},$$

$$y_1(t)=1-t^2,$$

$$y_2(t)=\frac{t^3(t+1)}{2},$$

and

$$\sum_{i=0}^{2} |y_i(t)| < 1 \qquad \text{unless } t \text{ is a node.}$$

Up to now, proofs of the special cases in which the Bernstein and Erdös conditions have been upheld are dependent upon methods similar to those used in Theorem 1 of this paper. The heart of this proof is the reduction of several $(n-1)\times(n-1)$ square matrices of partial derivatives

$$(\partial\lambda_i/\partial t_j)_{\substack{i=1,j=1\\ i\neq p}}^{n,n-1}, \qquad p\in\{1,\dots,n\}$$

to matrices whose entries are point evaluations of functions lying in an $n-1$-dimensional space spanned by a Tchebycheff system. Non-singularity of these matrices thus is equivalent to linear independence of the evaluated functions. For this procedure to be generalized, even in the case that the range space Y is the span of an extended Tchebycheff system, some method of matrix reduction must be developed which will work in more general cases. Moreover, the resulting matrix of point evaluations must be one with which positive results can be obtained. In Theorem 2 of this paper, this second problem arises and is overcome in the immediate context.

In the case of polynomials generated by $t^{k+1},\dots, t^{k+n}$, treated in this paper, which is the first application of the Bernstein and Erdös conjectures to a space which does not contain constants, there is the third difficulty that interpolation must be investigated first on the interval $[0, b]$, with an appropriate redefinition of the quantity λ_1, before the result can be generalized to an interval $[a, b]$, with $0<a<b$.

There are, of course, obvious analogues to all of the above if one is working in the context of periodic functions.

Statement of the Problem

We wish first of all to characterize optimal interpolation from $C[0, b]$ or from

$$C_0[0, b]=\{f/f\in C[0, b] \text{ and } f(0)=0\},$$

choosing the space Y to be

$$\langle t^{k+1},\dots, t^{k+n}\rangle \qquad \text{or} \qquad \langle 1, t^{k+1},\dots, t^{k+n}\rangle,$$

where n is an integer $n \geqslant 2$ and $k \geqslant 0$.

Restricting our attention first to

$$\langle t^{k+1},\dots, t^{k+n}\rangle,$$

we choose nodes $t_1,\dots, t_n$, with

$$0 < t_1 < \cdots < t_n = b.$$

Then for each $i \in \{1,\dots, n\}$, the ith *fundamental polynomial* is given by

$$y_i(t) := \frac{t^{k+1}}{t_i^{k+1}} \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{t - t_j}{t_i - t_j}.$$

One may then construct an *interpolating projection*

$$L: C[0, b] \to Y$$

by

$$Lf := \sum_{i=1}^{n} f(t_i)\, y_i \qquad \text{for} \quad f \in C[0, b].$$

One then has, setting $t_0 = 0$ for convenience,

$$\|L\| = \left\| \sum_{i=1}^{n} |y_i| \right\| = \max_{j \in \{1,\dots,n\}} \max_{t \in [t_{j-1}, t_j]} \sum_{i=1}^{n} |y_i(t)|.$$

For $j \in \{2,\dots, n\}$ we denote by λ_j the maximum of

$$\sum_{i=1}^{n} |y_i|$$

on the interval $[t_{j-1}, t_j]$, and for $j \in \{1,\dots, n\}$ we let X_j denote the polynomial in Y whose restriction to $[t_{j-1}, t_j]$ agrees with $\sum_{i=1}^{n} |y_i|$ on that interval. For $j \in \{2,\dots, n\}$, we may let T_j be the (unique) point in (t_{j-1}, t_j) at which λ_j is attained, noting that

$$X_j(T_j) = \lambda_j$$

and

$$X_j'(T_j) = 0 \qquad \text{for} \quad j \in \{2,\dots, n\}.$$

We may also let T_1 be the least positive root of X_1', and we define

$$\lambda_1 = X_1(T_1).$$

It is easy to see that $T_1 \in (0, T_2)$, and we have

$$\lambda_1 = \max_{t \in [0, t_1]} \sum_{i=1}^{n} |y_i| \qquad \text{whenever} \qquad 0 < T_1 \leqslant t_1.$$

On the other hand, if $t_1 < T_1$, this maximum is equal to 1 and is strictly less than λ_1, which remains in this case less than λ_2.

Interpolation into the same space of polynomials from $C[a, b]$, where a and b are positive, with nodes $t_1, \dots, t_n$ such that

$$a = t_1 < t_2 < \cdots < t_n = b$$

will be discussed after the treatment of interpolation on the interval $[0, b]$. The notation used will be that given here.

If one wishes to interpolate with the space

$$\langle 1, t^{k+1}, \dots, t^{k+n} \rangle$$

on the nodes $t_0, \dots, t_n$, with

$$0 = t_0 < t_1 < \cdots < t_n = b,$$

one has the same construction as before for the fundamental polynomials $y_1, \dots, y_n$. It is then possible to set

$$y_0 = 1 - \sum_{i=1}^{n} y_i,$$

and to define $X_1, \dots, X_n$, $\lambda_1, \dots, \lambda_n$, and $T_2, \dots, T_n$ in the same manner as before. Then T_1 also may be defined in a more standard manner as the unique point in $[t_0, t_1]$ which satisfies

$$X_1(T_1) = \lambda_1.$$

Optimization of Interpolation on $[0, b]$

We summarize our results as

THEOREM 1. *Let L be an interpolating projection into the space*

$$Y = \langle t^{k+1}, \dots, t^{k+n} \rangle,$$

where t is restricted to the closed interval $[0, b]$, and where n is an integer, $n \geqslant 2$, and $k > 0$. Then:

(i) *For L to be of minimal norm, it is necessary that $\lambda_1 = \cdots = \lambda_n$.*

(ii) *Equality of $\lambda_1, \dots, \lambda_n$ occurs at a unique set of nodes.*

(iii) *If one of $\lambda_1, \dots, \lambda_n$ is greater than the norm of the minimal projection, then another is less.*

(iv) *Results* (i), (ii), *and* (iii) *also hold if the space Y is taken to be*

$$\langle 1, t^{k+1}, \dots, t^{k+n} \rangle.$$

Remarks. (a) Clearly, (ii) implies the converse of (i), namely, that for L to be of minimal norm, it is sufficient that $\lambda_1 = \cdots = \lambda_n$.

(b) Norm of the optimal interpolating projection L does not depend upon the length of the interval $[0, b]$. This follows because of the fact that the natural isometry from $C[0, 1]$ (respectively $C_0[0, 1]$) to $C[0, b]$ (respectively $C_0[0, b]$) carries the space $\langle t^{k+1}, \dots, t^{k+n} \rangle$ into itself.

(c) More generally than in (b), one may state the following:
Let $F(X)$ be any normed linear space of functions defined on an underlying set X. Let X' be any set homeomorphic to X and $h: X \to X'$ any homeomorphism between the two sets. Then h induces an isometric isomorphism between $F(X)$ and $F(X')$, where $g \in F(X')$ if $g \circ h = f$ for some $f \in F(X)$. In particular, the natural homeomorphism $t \to 1/t$ between the intervals $[0, 1]$ and $[1, \infty]$ induces a natural isometry between $C_0[0, 1]$ and $C_0[1, \infty]$, the space of all functions continuous on $[1, \infty]$ whose value at ∞ is zero. Thus, results (i), (ii), and (iii) also hold for interpolation into the space

$$\langle 1/t^{k+1}, \dots, 1/t^{k+n} \rangle \qquad \text{restricted to } [1, \infty],$$

which is itself isomorphic to the space

$$\langle t^{k+1}, \dots, t^{k+n} \rangle \qquad \text{restricted to } [0, 1].$$

(d) The proof of Theorem 1(iv) mimics that of parts (i), (ii), and (iii) and will therefore not be given in separate form.

Proof of Theorem 1. Establishment of (i), (ii), and (iii) depends upon consideration of the function

$$(t_1, \dots, t_{n-1}) \to (\lambda_1, \dots, \lambda_n),$$

which is defined whenever $0 < t_1 < \cdots < t_n = b$, and its derivative

$$D = \begin{pmatrix} \partial\lambda_1/\partial t_1 & \cdots & \partial\lambda_n/\partial t_1 \\ & \cdots & \\ \partial\lambda_1/\partial t_{n-1} & \cdots & \partial\lambda_n/\partial t_{n-1} \end{pmatrix}.$$

which exists and is continuous on the same domain. The fundamentals of the proof of Theorem 1 are thus the same as those of previous, similar results on optimization of interpolation, including in particular those of [6] and [7]. Therefore, the proof is given here in outline, with appropriate care shown for the differences which our particular cases present.

We let D_p denote the determinant of the square matrix obtained by deletion of the ith column of D, for $i \in \{1,..., n\}$, after which one may establish conditions

(1) $D_p \neq 0$, globally and for $p \in \{1,..., n\}$, and

(2) $(-1)^i D_i/D_1 < 0$ for $p \in \{2,..., n\}$, which also holds globally. Conditions (1) and (2) will be discussed in more detail after the main body of the proof, which we now continue to present.

From (1), assertion (i) follows directly. For, if for some $p \in \{1,..., n\}$, $\lambda_p < \|L\|$, non-singularity of D_p implies that we may perturb $t_1,..., t_{n-1}$ to decrease simultaneously each of $\lambda_1,..., \lambda_n$ except λ_p.

From (2), we may deduce the remaining assertions. We conclude immediately that

$$\text{(3)} \qquad \det \left(\frac{\partial(\lambda_{p+1} - \lambda_p)}{\partial t_j}\right)_{j,p=1}^{n-1} = \sum_{p=1}^{n} (-1)^{p+1} D_p \neq 0$$

holds globally, from which it follows that the map

$$(t_1,..., t_{n-1}) \rightarrow (\lambda_2 - \lambda_1, \lambda_3 - \lambda_2,..., \lambda_n - \lambda_{n-1})$$

is a local homeomorphism. Following the arguments of [3], which apply here without essential change, we establish that (2) holds, and furthermore that our local homeomorphism is in fact global, from which (ii) follows as an immediate consequence.

Assertion (iii) is implied by Theorem 2 of [3]. That theorem, whose proof remains valid in the new context, states the slightly stronger assertion that, if $t_1,..., t_{n-1}$ and $s_1,..., s_{n-1}$ are such that

$$\lambda_i(t_1,..., t_{n-1}) \leqslant \lambda_i(s_1,..., s_{n-1}) \qquad \text{for} \quad i \in \{1,..., n\},$$

then

$$t_j = s_j \qquad \text{for} \quad j \in \{1,..., n-1\}.$$

This concludes the proof of Theorem 1.

A discussion of conditions (1) *and* (2). The matrix D may be reduced to an equivalent matrix by the following steps:

(a) $\partial\lambda_i/\partial t_j = -y_j(T_i) X_i'(t_j)$ for $i \in \{1,..., n\}$ and $j \in \{1,..., n-1\}$.

(b) For $j \in \{1,..., n-1\}$, the jth row of D is multiplied by the non-zero expression

$$t_j \prod_{\substack{l=1 \\ l \neq j}}^{n} (t_j - t_l),$$

cancelling simultaneously the multiple root at 0 of $X_i'(t_j)$ and the denominator of y_j.

(c) For $i \in \{1,..., n\}$, the ith column is divided by the non-zero expression

$$T_i^{k+1} \prod_{l=1}^{n} (T_i - t_l).$$

The matrix which results from these operations on D is an "evaluation matrix," each entry $\partial\lambda_i/\partial t_j$ being replaced by

$$\frac{X_i'(t_j)}{t_j^k(t_j - T_j)} = q_i(t_j),$$

where, for $i \in \{1,..., n\}$, q_i is a polynomial of degree $n-2$ or less. The proof of (1) and (2) thus depends upon a proof that the set $\{q_1,..., q_n\}$ of polynomials thus defined becomes a basis for the space of polynomials of degree $n-2$ or less whenever any one of $q_1,..., q_n$ is deleted from the set. The reader is referred to [3] or [7] for this proof, which depends upon a globally invariant interlacing of the roots of $q_1,..., q_n$.

Generalization of Theorem 1 to the Interval $[a, b]$

We now consider interpolation into the same space

$$\langle t^{k+1},..., t^{k+n} \rangle$$

on the interval $[a, b]$, where $0 < a < b$. By Remark (a) following Theorem 1, it is possible to assume that

$$0 < a = t_1 < \cdots < t_n = 1 = b.$$

We will show, using the notation already established, the following:

THEOREM 2. *Let a and nodes $t_1,..., t_n$ be given as above. Then*:

(i) *In order that the norm of interpolation into $\langle t^{k+1},..., t^{k+n} \rangle$ on $[a, b]$ be minimized, it is necessary that $\lambda_2 = \cdots = \lambda_n$.*

(ii) *Given that $t_1 = a$ and $t_n = b = 1$, the condition that $\lambda_2 = \cdots = \lambda_n$ determines $t_2,..., t_{n-1}$ uniquely.*

(iii) *If, for some* $i \in \{1,..., n\}$, λ_i *exceeds* $C(n, k, a)$, *by which we denote the norm of the minimal projection based on the interval* $[a, 1]$, *there is* $j \in \{1,..., n\}$ *such that* λ_j *is less than* $C(n, k, a)$.

(iv) *For* $0 < a < 1$, *the quantity* $C(n, k, a)$ *is a strictly decreasing function of* a, *and moreover*

$$\lim_{a \to 1} C(n, k, a) = C_{n-1},$$

where C_{n-1} *denotes the norm of optimal interpolation with polynomials of degree* $n - 1$ *or less, which is independent of one's choice of an interval.*

Remark. The minimization of $\max\{\lambda_2,..., \lambda_n\}$ with the nodes t_1 and t_n fixed is not an analogue of the problem dealt with in Theorem 1, and the original proof does not directly apply. The difficulty is as follows.

Assume that one wishes to vary $t_2,..., t_{n-1}$ in order to effect a decrease in all but one of $\lambda_2,..., \lambda_n$. As before, this is possible if a certain matrix whose entries are partial derivatives is globally non-singular. As before, this matrix (of dimension $(n-2) \times (n-2)$ this time) is equivalent to an "evaluation matrix" involving $n-2$ polynomials evaluated at $n-2$ points. Unfortunately, the polynomials are the same as those involved in the proof of Theorem 1, and their degree is $n-2$ or less, not $n-3$ or less. It does not usually follow that a linear combination of polynomials of degree $n-2$ or less which is zero at $n-2$ points is identically zero. However, it does follow in certain special cases, on one of which the following proof is based. Specifically, Lemma 9 of [7] states that no non-trivial linear combination of the polynomials $q_3,..., q_n$ can have roots at the nodes $t_2,..., t_{n-1}$, inasmuch as

$$T_2 < t_2 < T_3 < t_3 < \cdots < T_{n-1} < t_{n-1} < T_n.$$

Proof of Theorem 2. From the preceding remark, it follows that the matrix

$$\begin{pmatrix} \partial\lambda_3/\partial t_2 & \cdots & \partial\lambda_n/\partial t_2 \\ & \cdots & \\ \partial\lambda_3/\partial t_{n-1} & \cdots & \partial\lambda_n/\partial t_{n-1} \end{pmatrix}$$

is non-singular for $0 < t_1 < t_2 < \cdots < t_n$. From this fact it follows by the Implicit Function Theorem that, beginning at any initial position of the nodes $t_1,..., t_{n-1}$, it is possible to vary t_1 within some neighborhood and to move $t_2,..., t_{n-1}$ in such a way that $\lambda_3 \cdots \lambda_n$ retain their initial values. Moreover, as discussed in [7], the neighborhood upon which the implicit function

$$t_1 \to (t_2,..., t_{n-1})$$

is defined is in fact the whole interval $(0, 1)$. As t_1 approaches $t_n = 1$, the basis functions $y_1, ..., y_n$ uniformly and smoothly approach polynomials of degree $n-1$ on the interval $[t_1, t_n]$. It is also clear that, as t_1 increases, and $t_2, ..., t_{n-1}$ move subject to the condition that $\lambda_3, ..., \lambda_n$ remain constant, it is necessary that $\lambda_1 \to \infty$. By conditions (1) and (2) laid down in the proof of Theorem 1, λ_1 must in fact strictly increase, and moreover λ_2 must strictly decrease.

We are now in a position to demonstrate (i) of Theorem 2. Assume first of all that it is desired to decrease simultaneously $\lambda_3, ..., \lambda_n$. This can be done easily, because the non-singularity conditions given in the Remark above are precisely what is needed. On the other hand, it may be desired to decrease simultaneously all of $\lambda_2, ..., \lambda_n$ except for some λ_p, $p \in \{3, ..., n\}$. We may move $t_1, ..., t_{n-1}$, as described above, until they are in as small a neighborhood of 1 as desired, at which point the nodes $t_2, ..., t_{n-1}$ may be perturbed in such a way as to effect a decrease in λ_2, and an increase in λ_p, while the others do not change. We may now return the nodes to the original interval, moving $t_2, ..., t_{n-1}$ as functions of t_1, defined by the condition that $\lambda_3, ..., \lambda_n$ retain their new values. If λ_2 is now less than its original value, we have succeeded in reducing the problem to that of reducing simultaneously $\lambda_3, ..., \lambda_n$. If on the other hand the new value of λ_2 is exactly the same as or exceeds the old, we have a situation in which, for the same interval of interpolation, we have two different sets of nodes, say, $T = \{t_1, ..., t_n\}$ and $S = \{s_1, ..., s_n\}$, such that $t_1 = s_1$ and $t_n = s_n = 1$, and we also have

$$\lambda_i(T) \leqslant \lambda_i(S) \qquad \text{for} \quad i \in \{2, ..., n\}.$$

As will be shown in the course of proving (iii), this cannot be the case.

Parts (ii) and (iv) of Theorem 2 may be handled together. Assuming first of all that, for some $t_1, ..., t_{n-1}$, we have $\lambda_3 = \cdots = \lambda_n = c$, for some c, we note that, moving $t_1, ..., t_{n-1}$ as before subject to the condition that $\lambda_3 = \cdots = \lambda_n = c$, there is a unique position of $t_1, ..., t_{n-1}$ such that $\lambda_2 = c$ also. This fact defines t_1 as a function of c which is continuous, differentiable, and $dt_1/dc < 0$. Thus, c is also a function of t_1, and $\lim_{t_1 \to 1} c = C_{n-1}$, the norm of optimal interpolation with polynomials of degree $n-1$ or less.

We move to the proof of (iii). Theorem 2 of [2] was used in proving the corresponding hypothesis of Theorem 1. That theorem of [2] stated that, if

$$T = (t_0, ..., t_n)$$

and

$$S = (s_0, ..., s_n)$$

were systems of nodes with $0 = s_0 = t_0$ and $b = s_n = t_n$ such that

$$\lambda_i(T) \leqslant \lambda_i(S) \qquad \text{for all} \quad i \in \{1,..., n\},$$

then

$$T = S,$$

from which (iii) of Theorem 1 follows immediately. A strengthened version of this theorem is needed here, which states that, if $t_1 = s_1 = a$, and if

$$\lambda_i(T) \leqslant \lambda_i(S) \qquad \text{for} \quad i \in \{2,..., n\},$$

then

$$T = S.$$

Obviously, this strengthened version is true locally, that is, if the interval $[a, 1]$ is sufficiently short, because of the previously noted uniform convergence of the whole system of interpolation to a system of interpolation with ordinary polynomials of degree $n - 1$ as $a \to 1$.

First of all, we note that, if in fact

$$\lambda_i(T) = \lambda_i(S) \qquad \text{for} \quad i \in \{3,..., n\},$$

while

$$\lambda_2(T) \leqslant \lambda_2(S),$$

equality of S and T follows by the arguments used in the proof of (ii) and (iv). For the conditions

$$\lambda_i = c_i \qquad \text{for} \quad i \in \{3,..., n\},$$

where c_i are several constants, determine the nodes $t_2,..., t_{n-1}$ as implicit functions of t_1.

Finally, if T and S were such that

$$\lambda_2(T) = \lambda_2(S)$$

and

$$\lambda_i(T) = \lambda_i(S) \qquad \text{for all} \quad i \in \{3,..., n\},$$

except for one index, p, where

$$\lambda_p(T) < \lambda_p(S),$$

it would then be possible to obtain a set of nodes T' for which

$$\lambda_2(T') < \lambda_2(S),$$

while

$$\lambda_i(T') = \lambda_i(S) \qquad \text{for} \quad i \in \{3, \ldots, n\}.$$

But, as seen above, this is not possible.

This completes the proof of Theorem 2.

We remark in conclusion that the methods of proof employed in Theorem 2 could be used to prove similar results for interpolation carried out with trigonometric polynomials on an interval $[0, b]$.

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